Curtin University

Curtin University, Semester 1, 2022
ECON 4002 (Dr. Lei Pan)
Problem Set 2 Solution
Due Friday, May 27th at 5:00pm AWST

Question 1. **[30 marks] Ramsey model**

Consider a model of a representative agent who maximises utility equal to $\sum_{t=0}^{\infty} \beta^t \log(c_t)$, where c_t represents consumption at t and $0 < \beta < 1$. There is a capital technology that produces ak_t^γ at t from k_t units of capital created at $t-1$. The agent starts with a capital stock k_0. The output from capital is taxed at the rate τ, with $0 < \tau < 1$. The tax revenue is used to finance wasteful government spending.

(a) [10 marks] Write down the social planner's problem and find conditions that characterise the solution.

Answer The social planner's problem is given by:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

$$s.t. \qquad c_t + k_{t+1} = (1-\tau)ak_t^\gamma \tag{1}$$

$$k_0 \text{ is given, } k_{t+1} \geq 0, \ \forall t$$

where Eq.(1) is the resource constraint in the economy. F.O.C with respect to k_{t+1} is given by:

$$\beta^t \frac{1}{c_t} \frac{\partial c_t}{\partial k_{t+1}} + \beta^{t+1} \frac{1}{c_{t+1}} \frac{\partial c_{t+1}}{\partial k_{t+1}} = 0$$

$$\frac{1}{c_t}(-1) + \beta \frac{1}{c_{t+1}}(1-\tau)a\gamma k_{t+1}^{\gamma-1} = 0$$

$$\frac{1}{c_t} = \frac{\beta(1-\tau)a\gamma k_{t+1}^{\gamma-1}}{c_{t+1}}, \ \forall t \tag{2}$$

Substituting resource constraint into Eq.(2), can get:

$$\frac{1}{(1-\tau)ak_t^\gamma - k_{t+1}} = \frac{\beta(1-\tau)a\gamma k_{t+1}^{\gamma-1}}{(1-\tau)ak_{t+1}^\gamma - k_{t+2}} \tag{3}$$

(b) [15 marks] Verify that $k_{t+1} = sk_t^\gamma$ is a solution if s is a constant.

Answer Eq.(3) is the equation to which we apply the method of undetermined coefficients. Conjecture that

$$k_{t+1} = sk_t^\gamma \tag{4}$$

then

$$k_{t+2} = sk_{t+1}^\gamma \tag{5}$$

Substituting Eq.(5) into (3), we have:

$$
\begin{aligned}
\frac{1}{(1-\tau)ak_t^\gamma - k_{t+1}} = \frac{\beta(1-\tau)a\gamma k_{t+1}^{\gamma-1}}{(1-\tau)ak_{t+1}^\gamma - sk_{t+1}^\gamma} = \frac{\beta(1-\tau)a\gamma k_{t+1}^{\gamma-1}}{[(1-\tau)a - s]k_{t+1}^\gamma} &= \frac{\beta(1-\tau)a\gamma}{[(1-\tau)a-s]k_{t+1}} \\
\Rightarrow \beta(1-\tau)a\gamma[(1-\tau)ak_t^\gamma - k_{t+1}] &= [(1-\tau)a - s]k_{t+1} \\
\Rightarrow [\beta(1-\tau)a\gamma + (1-\tau)a - s]k_{t+1} &= \beta(1-\tau)a\gamma(1-\tau)ak_t^\gamma \\
\Rightarrow k_{t+1} &= \frac{\beta(1-\tau)a\gamma(1-\tau)a}{\beta(1-\tau)a\gamma + (1-\tau)a - s}k_t^\gamma
\end{aligned}
$$

By the conjecture, $k_{t+1} = sk_t^\gamma$, therefore

$$
\begin{aligned}
\frac{\beta(1-\tau)a\gamma(1-\tau)a}{\beta(1-\tau)a\gamma + (1-\tau)a - s} &= s \\
\Rightarrow s^2 - [\beta(1-\tau)a\gamma + (1-\tau)a]s + \beta(1-\tau)a\gamma(1-\tau)a &= 0 \\
\Rightarrow [s - \beta(1-\tau)a\gamma][s - (1-\tau)a] &= 0 \\
s = \beta(1-\tau)a\gamma \text{ or } s &= (1-\tau)a
\end{aligned}
$$

(c) [5 marks] Find s.

Answer Notice that $s = (1-\tau)a$ implies zero consumption, which is ruled out by the log utility form. Therefore, we discard this solution. Then

$$s = \beta(1-\tau)a\gamma$$

and hence the policy function is given by:

$$k_{t+1} = \beta(1-\tau)a\gamma k_t^\gamma$$

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Question 2. **[40 marks] Ramsey model with endogenous labour**

Consider a representative household who lives forever and is endowed with one unit of time every period which she spends on leisure, l_t, and work, h_t. This household derives utility from consumption and leisure which is given by:

$$\sum_{t=0}^{\infty} \beta^t [\log(c_t) + \gamma \log(l_t)]$$

where $0 < \beta < 1$ is the discount rate. There is a constant return to scale production technology that allows firms to produce goods given by $F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}$ from K units of capital and H hours of labour. Firms must rent their capital and labour from households at the respective rental rates. The household owns all capital, there is full depreciation of capital and there is no labour augmenting neither productivity nor population growth.

(a) [15 marks] Write down the representative household's budget constraint in a given period t, and formulate her utility maximisation problem. Find the equation that determines the consumption path.

Answer The budget constraint of an individual is:

$$c_t + s_t = w_t h_t + r_t s_{t-1}$$

Therefore, household optimisation problem is:

$$\max_{s_t, l_t} \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \gamma \log(l_t)]$$
$$s.t. \qquad c_t + s_t = w_t h_t + r_t s_{t-1}$$
$$l_t = 1 - h_t$$

Or

$$\max_{s_t, h_t} \sum_{t=0}^{\infty} \beta^t [\log(w_t h_t + r_t s_{t-1} - s_t) + \gamma \log(1 - h_t)]$$

Therefore, the FOCs are

$$\frac{\partial U}{\partial s_t} = \beta^t \frac{-1}{w_t h_t + r_t s_{t-1} - s_t} + \beta^{t+1} \frac{r_{t+1}}{w_{t+1} h_{t+1} + r_{t+1} s_t - s_{t+1}} = 0$$

and

$$\frac{\partial U}{\partial h_t} = \beta^t \frac{w_t}{w_t h_t + r_t s_{t-1} - s_t} + \beta^t \frac{-\gamma}{1 - h_t} = 0$$

Therefore, we have:

$$\frac{1}{w_t h_t + r_t s_{t-1} - s_t} = \frac{\beta r_{t+1}}{w_{t+1} h_{t+1} + r_{t+1} s_t - s_{t+1}}$$
$$w_t(1 - h_t) = \gamma(w_t h_t + r_t s_{t-1} - s_t)$$

The first equation is Euler equation when we can write it as $\frac{c_{t+1}}{c_t} = \beta r_{t+1}$.

(b) [10 marks] Write down conditions that characterise firm's profit maximisation problem and market clearing conditions for the capital and labour market.

Answer Firm's optimisation problem is

$$\max_{H_t, K_t} K_t^\alpha H_t^{1-\alpha} - w_t H_t - r_t K_t$$

Thus,

$$\frac{\partial \pi_t}{\partial H_t} = (1-\alpha) K_t^\alpha H_t^{-\alpha} - w_t = 0 \Rightarrow (1-\alpha)(\frac{K_t}{H_t})^\alpha = w_t$$
$$\frac{\partial \pi_t}{\partial K_t} = \alpha K_t^{\alpha-1} H_t^{1-\alpha} - r_t = 0 \Rightarrow \alpha(\frac{K_t}{H_t})^{\alpha-1} = r_t$$

Or,

$$w_t = (1-\alpha)(\frac{K_t}{H_t})^\alpha = (1-\alpha)\frac{Y_t}{H_t}$$
$$r_t = \alpha(\frac{K_t}{H_t})^{\alpha-1} = \alpha\frac{Y_t}{K_t}$$

The marking clearing conditions are:

$$K_t = L s_{t-1}$$
$$H_t = L h_t$$

where L is normalised to one in the representative agent framework. Recall that in the Cobb-Douglas production function we have:

$$w_t H_t + r_t K_t = Y_t$$

(c) [15 marks] Derive the per capita capital accumulation for this economy. What is the steady state per capita capital and consumption in this economy?

Answer Go back to two FOCs from household optimisation problem

$$\frac{1}{Y_t - K_{t+1}} = \frac{\beta\alpha(\frac{Y_{t+1}}{K_{t+1}})}{Y_{t+1} - K_{t+2}} \tag{6}$$
$$(1-\alpha)\frac{Y_t}{H_t}(1 - H_t) = \gamma(K_t^\alpha H_t^{1-\alpha} - K_{t+1}) \tag{7}$$

Thus, in the steady state Eq.(6) becomes

$$\frac{Y}{K} = \frac{1}{\beta\alpha} \tag{8}$$

Dividing both sides of Eq.(7) by K_t, we get

$$(1-\alpha)\frac{Y_t}{K_t}\frac{1-H_t}{H_t} = \gamma(\frac{K_t^{\alpha}H_t^{1-\alpha}}{K_t} - \frac{K_{t+1}}{K_t})$$

Then, in the steady state, the above equation becomes

$$(1-\alpha)\frac{Y}{K}\frac{1-H}{H} = \gamma(\frac{Y}{K} - 1) \tag{9}$$

Plug Eq.(8) into (9), we have

$$\frac{1-H}{H} = \frac{\gamma(\frac{1}{\beta\alpha} - 1)}{(1-\alpha)\frac{1}{\beta\alpha}} \tag{10}$$

Therefore,

$$H = \frac{1}{1+\frac{\gamma(\frac{1}{\beta\alpha}-1)}{(1-\alpha)\frac{1}{\beta\alpha}}} = \frac{(1-\alpha)\frac{1}{\beta\alpha}}{(1-\alpha+\gamma)\frac{1}{\beta\alpha} - \gamma} = \frac{1-\alpha}{(1-\alpha+\gamma) - \gamma\beta\alpha} \tag{11}$$

Total output in the steady state is $Y = K^{\alpha}H^{1-\alpha}$ and from Eq.(8) we know $Y = \frac{1}{\beta\alpha}K$, we therefore have

$$\frac{1}{\beta\alpha}K = K^{\alpha}H^{1-\alpha}$$

We can solve the above equation for K given H found in Eq.(11).

$$K^{1-\alpha} = \frac{H^{1-\alpha}}{\frac{1}{\beta\alpha}}$$

$$K = (\frac{H^{1-\alpha}}{\frac{1}{\beta\alpha}})^{\frac{1}{1-\alpha}} \tag{12}$$

After considering market clearing conditions, in the steady state consumption becomes $c = Y - K$, which equals to

$$c = K(\frac{Y}{K} - 1)$$

From Eq.(8) we have $\frac{Y}{K}$ and from Eq.(12) we have K.

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Question 3. **[30 marks] Lucas (1988) human capital accumulation model**

A representative agent has 1 unit of time that he can allocate to produce consumption goods and accumulate education (human capital). The periodic utility function is given by $u(C_t) = \frac{C_t^{1-\theta}-1}{1-\theta}$, where $\theta > 1$ and the production function is given by $Y_t = K_t^{\alpha}(\phi_t H_t)^{1-\alpha}$ where H_t denotes human capital, ϕ_t is the fraction of hours devoted to work and $1-\phi_t$ is the fraction devoted to education. New human capital is produced using the CRS technology

$$H_{t+1} = B(1-\phi_t)H_t$$

where $B > 0$ measures the return to education, and $H_0 > 0$ is given.

(a) [5 marks] Write the aggregate resource constraint.

Answer The aggregate resource constraint is: $C_t + K_{t+1} = K_t^{\alpha}(\phi_t H_t)^{1-\alpha} + (1-\delta)K_t$

(b) [25 marks] Solve the social planner's problem and find the optimal allocation of time between human capital and work. Show there is a balanced growth equilibrium where C_t, K_t, Y_t and H_t all grow at an identical rate.

Answer As in Ramsey model, we can find the competitive equilibrium from solving a social planner's problem:

$$\max_{\{\phi_t, H_{t+1}, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta}-1}{1-\theta}$$

$$s.t. \qquad C_t + K_{t+1} = K_t^{\alpha}(\phi_t H_t)^{1-\alpha} + (1-\delta)K_t \tag{13}$$

$$H_{t+1} = B(1-\phi_t)H_t \tag{14}$$

where K_0 and H_0 are given. The Lagrangian equation for this optimisation problem can be written as:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ \frac{[K_t^{\alpha}(\phi_t H_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}]^{1-\theta} - 1}{1-\theta} + \lambda_t [B(1-\phi_t)H_t - H_{t+1}] \}$$

Extend Lagrangian equation is

$$\mathcal{L} = ... + \beta^t \{ \frac{[K_t^{\alpha}(\phi_t H_t)^{1-\alpha} + (1-\delta)K_t - K_{t+1}]^{1-\theta} - 1}{1-\theta} + \lambda_t [B(1-\phi_t)H_t - H_{t+1}] \}$$
$$+ \beta^{t+1} \{ \frac{[K_{t+1}^{\alpha}(\phi_{t+1} H_{t+1})^{1-\alpha} + (1-\delta)K_{t+1} - K_{t+2}]^{1-\theta} - 1}{1-\theta} + \lambda_{t+1} [B(1-\phi_{t+1})H_{t+1} - H_{t+2}] \} ...$$

F.O.C w.r.t. ϕ_t:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi_t} &= \beta^t\{C^{-\theta}[(1-\alpha)K_t^\alpha \phi_t^{-\alpha} H_t^{1-\alpha}] + \lambda_t(-BH_t)\} \\ &= \beta^t\{C^{-\theta}[\frac{\phi_t}{\phi_t}(1-\alpha)K_t^\alpha \phi_t^{-\alpha} H_t^{1-\alpha}] - \lambda_t BH_t\} \\ &= \beta^t\{C^{-\theta}[\frac{(1-\alpha)K_t^\alpha \phi_t^{1-\alpha} H_t^{1-\alpha}}{\phi_t}] - \lambda_t BH_t\} \\ &= \beta^t\{C^{-\theta}[\frac{(1-\alpha)Y_t}{\phi_t}] - \lambda_t BH_t\}\end{aligned}$$

F.O.C w.r.t. H_{t+1}:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial H_{t+1}} &= \beta^t\lambda_t(-1) + \beta^{t+1}\{C_{t+1}^{-\theta}(1-\alpha)K_{t+1}^\alpha \phi_{t+1}^{1-\alpha} H_{t+1}^{-\alpha} + \lambda_{t+1}B(1-\phi_{t+1})\} = 0 \\ &= -\lambda_t + \beta\{C_{t+1}^{-\theta}(1-\alpha)K_{t+1}^\alpha \phi_{t+1}^{1-\alpha} H_{t+1}^{-\alpha}\frac{H_{t+1}}{H_{t+1}} + \lambda_{t+1}B(1-\phi_{t+1})\} = 0 \\ &= -\lambda_t + \beta\{C_{t+1}^{-\theta}\frac{(1-\alpha)Y_{t+1}}{H_{t+1}} + \lambda_{t+1}B(1-\phi_{t+1})\} = 0\end{aligned}$$

F.O.C w.r.t. K_{t+1}:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= \beta^t C_t^{-\theta}(-1) + \beta^{t+1}C_{t+1}^{-\theta}[\alpha K_{t+1}^{\alpha-1}(\phi_{t+1}H_{t+1})^{1-\alpha} + (1-\delta)] = 0 \\ &= -C_t^{-\theta} + \beta C_{t+1}^{-\theta}[\alpha K_{t+1}^{\alpha-1}(\phi_{t+1}H_{t+1})^{1-\alpha}\frac{K_{t+1}}{K_{t+1}} + (1-\delta)] = 0 \\ &= -C_t^{-\theta} + \beta C_{t+1}^{-\theta}[\alpha\frac{Y_{t+1}}{K_{t+1}} + (1-\delta)] = 0\end{aligned}$$

To summarise, the F.O.C.s are:

$$\phi_t : C_t^{-\theta}[\frac{(1-\alpha)Y_t}{\phi_t}] = \lambda_t BH_t \tag{15}$$

$$H_{t+1} : \lambda_t = \beta\{C_{t+1}^{-\theta}[\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}] + \lambda_{t+1}B(1-\phi_{t+1})\} \tag{16}$$

$$K_{t+1} : C_t^{-\theta} = \beta\{C_{t+1}^{-\theta}[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)]\} \tag{17}$$

From Eq.(15), we find λ_t

$$\lambda_t = \frac{C_t^{-\theta}[\frac{(1-\alpha)Y_t}{\phi_t}]}{BH_t}$$

and substitute it into Eq.(16), we get:

$$\begin{aligned}
\frac{C_t^{-\theta}[\frac{(1-\alpha)Y_t}{\phi_t}]}{BH_t} &= \beta\{C_{t+1}^{-\theta}[\frac{(1-\alpha)Y_{t+1}}{H_{t+1}}] + \frac{C_{t+1}^{-\theta}[\frac{(1-\alpha)Y_{t+1}}{\phi_{t+1}}]}{BH_{t+1}}B(1-\phi_{t+1})\} \\
\frac{C_t^{-\theta}Y_t}{BH_t\phi_t} &= \beta C_{t+1}^{-\theta}\{\frac{Y_{t+1}}{H_{t+1}} + \frac{Y_{t+1}(1-\phi_{t+1})}{H_{t+1}\phi_{t+1}}\} \\
\frac{C_t^{-\theta}}{C_{t+1}^{-\theta}} &= \beta B\frac{Y_{t+1}H_t}{Y_tH_{t+1}}\phi_t\{1 + \frac{(1-\phi_{t+1})}{\phi_{t+1}}\} \\
(\frac{C_{t+1}}{C_t})^{\theta} &= \beta B\{\frac{Y_{t+1}}{Y_t}\frac{H_t}{H_{t+1}}\frac{\phi_t}{\phi_{t+1}}\} \qquad (18)
\end{aligned}$$

while Eq.(17) can be written as:

$$(\frac{C_{t+1}}{C_t})^{\theta} = \beta[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)] \qquad (19)$$

Finding growth rate, start from Eq.(14)

$$\frac{H_{t+1}}{H_t} = B(1-\phi_t)$$

Hence,

$$g = \frac{H_{t+1}-H_t}{H_t} = B(1-\phi_t) - 1 \qquad (20)$$

There is a balanced growth equilibrium where C_t, K_t, Y_t and H_t all grow at an identical constant rate. Look at Eq.(18) when C, Y and H all grow at a same rate g.

$$(1+g)^{\theta} = \beta B\{(1+g)\frac{1}{1+g}\frac{\phi}{\phi}\}$$

Therefore, we can find g

$$g = (\beta B)^{\frac{1}{\theta}} - 1$$

From Eq.(20), we also have $g = B(1-\phi^*) - 1$, we therefore can solve for ϕ^*.

- C_t, K_t, Y_t and H_t all grow at the same rate as human capital accumulation, $g = B(1-\phi^*) - 1 = (\beta B)^{\frac{1}{\theta}} - 1$, where $\phi^* = 1 - \frac{(\beta B)^{\frac{1}{\theta}}}{B}$ is the optimal allocation of individual's time between production and education.
- Therefore, we get positive endogenous growth provided $\beta B > 1$. To get this work, the returns to educations B have to be sufficiently high.

The End